Final Exam - Harmonic Analysis (Elective)
B. Math III

30 April, 2024

(i) Duration of the exam is 3 hours.

(ii) The maximum number of points you can score in the exam is 100 (total = 110).

(iii) You are not allowed to consult any notes or external sources for the exam.

Name: ____________________

Roll Number: ____________________

1. (15 points) Let $G = GL_n(\mathbb{R})$ be the group of invertible $n \times n$ matrices under matrix multiplication. Show that a left-invariant Haar measure on G is given by $dx/|\det x|$, if dx is a Haar measure on the n^2-dimensional space of all $n \times n$ matrices under matrix addition.

Total for Question 1: 15

2. Let $\mathbb{R}^\times$ denote the group of non-zero real numbers under multiplication. Let G be the group of all affine maps $x \mapsto ax + b$ with $a \in \mathbb{R}^\times$ and $b \in \mathbb{R}$.
   (a) (5 points) Show that G is locally compact.
   (b) (5 points) Compute the left-invariant Haar measure of G.
   (c) (5 points) Compute the right-invariant Haar measure of G.
   (d) (5 points) Compute the modular function of G.

Total for Question 2: 20

3. (15 points) Let A be a unital Banach algebra and $x, y \in A$. Prove that $xy - yx \neq 1$. In other words, the Heisenberg commutation relation cannot be realized in Banach algebras.

Total for Question 3: 15

4. (20 points) Let G be a locally compact (not necessarily abelian) group. Show that $\widehat{G}$ is an LCA group, and that the assignment $G \mapsto \widehat{G}$ is a contravariant functor, in the sense that for any continuous homomorphism $\varphi : G' \to G$, there is a dual continuous homomorphism $\widehat{\varphi} : \widehat{G} \to \widehat{G'}$ given by $\widehat{\varphi}(\chi) = \chi \circ \varphi$.

Total for Question 4: 20

5. Let A be an LCA group, and let $f \in L^1(A)$.
   (a) (10 points) Show that $\widehat{f} \in C_0(\widehat{A})$, that is, $\widehat{f}$ vanishes at infinity on $\widehat{A}$.
   (b) (10 points) Show that the Fourier transform $L^1(A) \to C_0(\widehat{A})$ is injective.

Total for Question 5: 20

6. (20 points) Let $A = K \times \mathbb{R}^n \times \mathbb{Z}^m$ where K is a compact abelian group and $m, n \geq 0$. Prove that the mapping $x \mapsto \delta_x$ from A to $\widehat{\widehat{A}}$ where $\delta_x(\chi) = \chi(x)$ is a homeomorphism. (Prove this from first principles and not simply invoke Pontryagin duality theorem.)

Total for Question 6: 20